

Solving linear and quadratic simultaneous equations

A LEVEL LINKS

Scheme of work: 1c. Equations – quadratic/linear simultaneous

Key points

- Make one of the unknowns the subject of the linear equation (rearranging where necessary).
- Use the linear equation to substitute into the quadratic equation.
- There are usually two pairs of solutions.

Examples

Example 1 Solve the simultaneous equations $y = x + 1$ and $x^2 + y^2 = 13$

$x^2 + (x + 1)^2 = 13$ $x^2 + x^2 + x + x + 1 = 13$ $2x^2 + 2x + 1 = 13$ $2x^2 + 2x - 12 = 0$ $(2x - 4)(x + 3) = 0$ So $x = 2$ or $x = -3$ Using $y = x + 1$ When $x = 2$, $y = 2 + 1 = 3$ When $x = -3$, $y = -3 + 1 = -2$ So the solutions are $x = 2, y = 3$ and $x = -3, y = -2$ Check: equation 1: $3 = 2 + 1$ YES and $-2 = -3 + 1$ YES equation 2: $2^2 + 3^2 = 13$ YES and $(-3)^2 + (-2)^2 = 13$ YES	<ol style="list-style-type: none"> 1 Substitute $x + 1$ for y into the second equation. 2 Expand the brackets and simplify. 3 Factorise the quadratic equation. 4 Work out the values of x. 5 To find the value of y, substitute both values of x into one of the original equations. 6 Substitute both pairs of values of x and y into both equations to check your answers.
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Example 2 Solve $2x + 3y = 5$ and $2y^2 + xy = 12$ simultaneously.

$x = \frac{5-3y}{2}$ $2y^2 + \left(\frac{5-3y}{2}\right)y = 12$ $2y^2 + \frac{5y-3y^2}{2} = 12$ $4y^2 + 5y - 3y^2 = 24$ $y^2 + 5y - 24 = 0$ $(y+8)(y-3) = 0$ <p>So $y = -8$ or $y = 3$</p> <p>Using $2x + 3y = 5$ When $y = -8$, $2x + 3 \times (-8) = 5$, $x = 14.5$ When $y = 3$, $2x + 3 \times 3 = 5$, $x = -2$</p> <p>So the solutions are $x = 14.5$, $y = -8$ and $x = -2$, $y = 3$</p> <p>Check: equation 1: $2 \times 14.5 + 3 \times (-8) = 5$ YES and $2 \times (-2) + 3 \times 3 = 5$ YES equation 2: $2 \times (-8)^2 + 14.5 \times (-8) = 12$ YES and $2 \times (3)^2 + (-2) \times 3 = 12$ YES</p>	<p>1 Rearrange the first equation.</p> <p>2 Substitute $\frac{5-3y}{2}$ for x into the second equation. Notice how it is easier to substitute for x than for y.</p> <p>3 Expand the brackets and simplify.</p> <p>4 Factorise the quadratic equation.</p> <p>5 Work out the values of y.</p> <p>6 To find the value of x, substitute both values of y into one of the original equations.</p> <p>7 Substitute both pairs of values of x and y into both equations to check your answers.</p>
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Practice

Solve these simultaneous equations.

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|---------------------------------------------|---------------------------------------------|
| 1 $y = 2x + 1$
$x^2 + y^2 = 10$ | 2 $y = 6 - x$
$x^2 + y^2 = 20$ |
| 3 $y = x - 3$
$x^2 + y^2 = 5$ | 4 $y = 9 - 2x$
$x^2 + y^2 = 17$ |
| 5 $y = 3x - 5$
$y = x^2 - 2x + 1$ | 6 $y = x - 5$
$y = x^2 - 5x - 12$ |
| 7 $y = x + 5$
$x^2 + y^2 = 25$ | 8 $y = 2x - 1$
$x^2 + xy = 24$ |
| 9 $y = 2x$
$y^2 - xy = 8$ | 10 $2x + y = 11$
$xy = 15$ |

Extend

- | | |
|------------------------------------------|-----------------------------------------|
| 11 $x - y = 1$
$x^2 + y^2 = 3$ | 12 $y - x = 2$
$x^2 + xy = 3$ |
|------------------------------------------|-----------------------------------------|

Answers

1 $x = 1, y = 3$

$$x = -\frac{9}{5}, y = -\frac{13}{5}$$

2 $x = 2, y = 4$

$$x = 4, y = 2$$

3 $x = 1, y = -2$

$$x = 2, y = -1$$

4 $x = 4, y = 1$

$$x = \frac{16}{5}, y = \frac{13}{5}$$

5 $x = 3, y = 4$

$$x = 2, y = 1$$

6 $x = 7, y = 2$

$$x = -1, y = -6$$

7 $x = 0, y = 5$

$$x = -5, y = 0$$

8 $x = -\frac{8}{3}, y = -\frac{19}{3}$

$$x = 3, y = 5$$

9 $x = -2, y = -4$

$$x = 2, y = 4$$

10 $x = \frac{5}{2}, y = 6$

$$x = 3, y = 5$$

11 $x = \frac{1+\sqrt{5}}{2}, y = \frac{-1+\sqrt{5}}{2}$

$$x = \frac{1-\sqrt{5}}{2}, y = \frac{-1-\sqrt{5}}{2}$$

12 $x = \frac{-1+\sqrt{7}}{2}, y = \frac{3+\sqrt{7}}{2}$

$$x = \frac{-1-\sqrt{7}}{2}, y = \frac{3-\sqrt{7}}{2}$$

Linear inequalities

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Scheme of work: 1d. Inequalities – linear and quadratic (including graphical solutions)

Key points

- Solving linear inequalities uses similar methods to those for solving linear equations.
- When you multiply or divide an inequality by a negative number you need to reverse the inequality sign, e.g. $<$ becomes $>$.

Examples

Example 1 Solve $-8 \leq 4x < 16$

$\begin{aligned} -8 &\leq 4x < 16 \\ -2 &\leq x < 4 \end{aligned}$	Divide all three terms by 4.
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Example 2 Solve $4 \leq 5x < 10$

$\begin{aligned} 4 &\leq 5x < 10 \\ \frac{4}{5} &\leq x < 2 \end{aligned}$	Divide all three terms by 5.
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Example 3 Solve $2x - 5 < 7$

$\begin{aligned} 2x - 5 &< 7 \\ 2x &< 12 \\ x &< 6 \end{aligned}$	<ol style="list-style-type: none"> Add 5 to both sides. Divide both sides by 2.
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Example 4 Solve $2 - 5x \geq -8$

$\begin{aligned} 2 - 5x &\geq -8 \\ -5x &\geq -10 \\ x &\leq 2 \end{aligned}$	<ol style="list-style-type: none"> Subtract 2 from both sides. Divide both sides by -5. Remember to reverse the inequality when dividing by a negative number.
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Example 5 Solve $4(x - 2) > 3(9 - x)$

$\begin{aligned} 4(x - 2) &> 3(9 - x) \\ 4x - 8 &> 27 - 3x \\ 7x - 8 &> 27 \\ 7x &> 35 \\ x &> 5 \end{aligned}$	<ol style="list-style-type: none"> Expand the brackets. Add $3x$ to both sides. Add 8 to both sides. Divide both sides by 7.
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Practice

1 Solve these inequalities.

a $4x > 16$

b $5x - 7 \leq 3$

c $1 \geq 3x + 4$

d $5 - 2x < 12$

e $\frac{x}{2} \geq 5$

f $8 < 3 - \frac{x}{3}$

2 Solve these inequalities.

a $\frac{x}{5} < -4$

b $10 \geq 2x + 3$

c $7 - 3x > -5$

3 Solve

a $2 - 4x \geq 18$

b $3 \leq 7x + 10 < 45$

c $6 - 2x \geq 4$

d $4x + 17 < 2 - x$

e $4 - 5x < -3x$

f $-4x \geq 24$

4 Solve these inequalities.

a $3t + 1 < t + 6$

b $2(3n - 1) \geq n + 5$

5 Solve.

a $3(2 - x) > 2(4 - x) + 4$

b $5(4 - x) > 3(5 - x) + 2$

Extend

6 Find the set of values of x for which $2x + 1 > 11$ and $4x - 2 > 16 - 2x$.

Answers

1 a $x > 4$

d $x > -\frac{7}{2}$

b $x \leq 2$

e $x \geq 10$

c $x \leq -1$

f $x < -15$

2 a $x < -20$

b $x \leq 3.5$

c $x < 4$

3 a $x \leq -4$

d $x < -3$

b $-1 \leq x < 5$

e $x > 2$

c $x \leq 1$

f $x \leq -6$

4 a $t < \frac{5}{2}$

b $n \geq \frac{7}{5}$

5 a $x < -6$

b $x < \frac{3}{2}$

6 $x > 5$ (which also satisfies $x > 3$)

Quadratic inequalities

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Scheme of work: 1d. Inequalities – linear and quadratic (including graphical solutions)

Key points

- First replace the inequality sign by = and solve the quadratic equation.
- Sketch the graph of the quadratic function.
- Use the graph to find the values which satisfy the quadratic inequality.

Examples

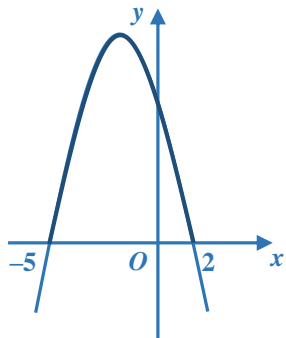
Example 1 Find the set of values of x which satisfy $x^2 + 5x + 6 > 0$

$x^2 + 5x + 6 = 0$ $(x + 3)(x + 2) = 0$ $x = -3 \text{ or } x = -2$ $x < -3 \text{ or } x > -2$	<ol style="list-style-type: none"> 1 Solve the quadratic equation by factorising. 2 Sketch the graph of $y = (x + 3)(x + 2)$ 3 Identify on the graph where $x^2 + 5x + 6 > 0$, i.e. where $y > 0$ 4 Write down the values which satisfy the inequality $x^2 + 5x + 6 > 0$
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Example 2 Find the set of values of x which satisfy $x^2 - 5x \leq 0$

$x^2 - 5x = 0$ $x(x - 5) = 0$ $x = 0 \text{ or } x = 5$ $0 \leq x \leq 5$	<ol style="list-style-type: none"> 1 Solve the quadratic equation by factorising. 2 Sketch the graph of $y = x(x - 5)$ 3 Identify on the graph where $x^2 - 5x \leq 0$, i.e. where $y \leq 0$ 4 Write down the values which satisfy the inequality $x^2 - 5x \leq 0$
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Example 3 Find the set of values of x which satisfy $-x^2 - 3x + 10 \geq 0$

$-x^2 - 3x + 10 = 0$ $(-x + 2)(x + 5) = 0$ $x = 2 \text{ or } x = -5$  $-5 \leq x \leq 2$	<ol style="list-style-type: none"> 1 Solve the quadratic equation by factorising. 2 Sketch the graph of $y = (-x + 2)(x + 5) = 0$ 3 Identify on the graph where $-x^2 - 3x + 10 \geq 0$, i.e. where $y \geq 0$ 3 Write down the values which satisfy the inequality $-x^2 - 3x + 10 \geq 0$
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Practice

- 1** Find the set of values of x for which $(x + 7)(x - 4) \leq 0$
- 2** Find the set of values of x for which $x^2 - 4x - 12 \geq 0$
- 3** Find the set of values of x for which $2x^2 - 7x + 3 < 0$
- 4** Find the set of values of x for which $4x^2 + 4x - 3 > 0$
- 5** Find the set of values of x for which $12 + x - x^2 \geq 0$

Extend

Find the set of values which satisfy the following inequalities.

- 6** $x^2 + x \leq 6$
- 7** $x(2x - 9) < -10$
- 8** $6x^2 \geq 15 + x$

Answers

1 $-7 \leq x \leq 4$

2 $x \leq -2$ or $x \geq 6$

3 $\frac{1}{2} < x < 3$

4 $x < -\frac{3}{2}$ or $x > \frac{1}{2}$

5 $-3 \leq x \leq 4$

6 $-3 \leq x \leq 2$

7 $2 < x < 2\frac{1}{2}$

8 $x \leq -\frac{3}{2}$ or $x \geq \frac{5}{3}$

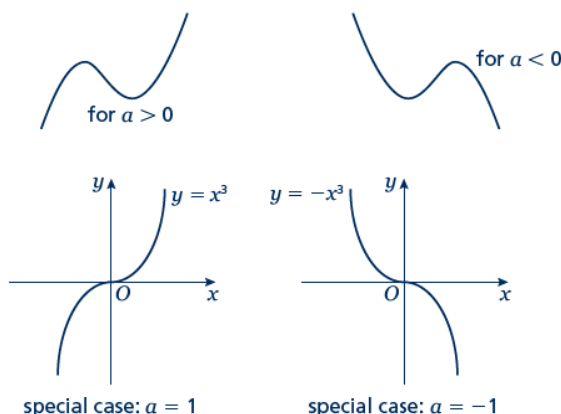
Sketching cubic and reciprocal graphs

A LEVEL LINKS

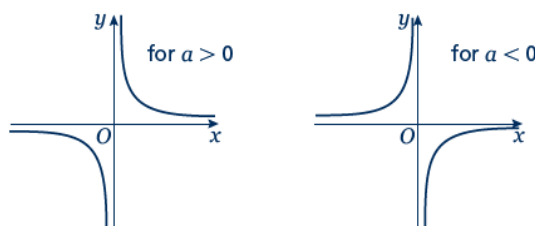
Scheme of work: 1e. Graphs – cubic, quartic and reciprocal

Key points

- The graph of a cubic function, which can be written in the form $y = ax^3 + bx^2 + cx + d$, where $a \neq 0$, has one of the shapes shown here.



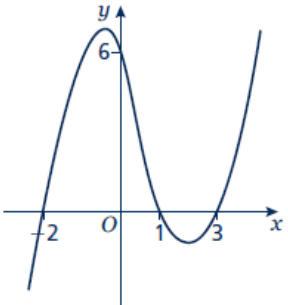

- The graph of a reciprocal function of the form $y = \frac{a}{x}$ has one of the shapes shown here.



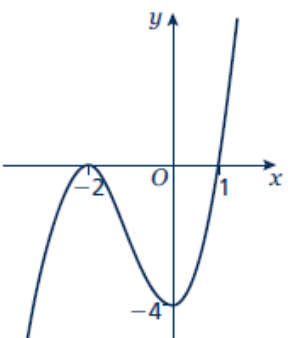

- To sketch the graph of a function, find the points where the graph intersects the axes.
- To find where the curve intersects the y -axis substitute $x = 0$ into the function.
- To find where the curve intersects the x -axis substitute $y = 0$ into the function.
- Where appropriate, mark and label the asymptotes on the graph.
- Asymptotes are lines (usually horizontal or vertical) which the curve gets closer to but never touches or crosses. Asymptotes usually occur with reciprocal functions. For example, the asymptotes for the graph of $y = \frac{a}{x}$ are the two axes (the lines $y = 0$ and $x = 0$).
- At the turning points of a graph the gradient of the curve is 0 and any tangents to the curve at these points are horizontal.
- A double root is when two of the solutions are equal. For example $(x - 3)^2(x + 2)$ has a double root at $x = 3$.
- When there is a double root, this is one of the turning points of a cubic function.

Examples

Example 1 Sketch the graph of $y = (x - 3)(x - 1)(x + 2)$

To sketch a cubic curve find intersects with both axes and use the key points above for the correct shape.	
<p>When $x = 0$, $y = (0 - 3)(0 - 1)(0 + 2)$ $= (-3) \times (-1) \times 2 = 6$ The graph intersects the y-axis at $(0, 6)$</p> <p>When $y = 0$, $(x - 3)(x - 1)(x + 2) = 0$ So $x = 3$, $x = 1$ or $x = -2$ The graph intersects the x-axis at $(-2, 0)$, $(1, 0)$ and $(3, 0)$</p> 	<ol style="list-style-type: none"> Find where the graph intersects the axes by substituting $x = 0$ and $y = 0$. Make sure you get the coordinates the right way around, (x, y). Solve the equation by solving $x - 3 = 0$, $x - 1 = 0$ and $x + 2 = 0$ Sketch the graph. $a = 1 > 0$ so the graph has the shape: 

Example 2 Sketch the graph of $y = (x + 2)^2(x - 1)$

To sketch a cubic curve find intersects with both axes and use the key points above for the correct shape.	
<p>When $x = 0$, $y = (0 + 2)^2(0 - 1)$ $= 2^2 \times (-1) = -4$ The graph intersects the y-axis at $(0, -4)$</p> <p>When $y = 0$, $(x + 2)^2(x - 1) = 0$ So $x = -2$ or $x = 1$</p> <p>$(-2, 0)$ is a turning point as $x = -2$ is a double root. The graph crosses the x-axis at $(1, 0)$</p> 	<ol style="list-style-type: none"> Find where the graph intersects the axes by substituting $x = 0$ and $y = 0$. Solve the equation by solving $x + 2 = 0$ and $x - 1 = 0$ $a = 1 > 0$ so the graph has the shape: 

Practice

1 Here are six equations.

A $y = \frac{5}{x}$

B $y = x^2 + 3x - 10$

C $y = x^3 + 3x^2$

D $y = 1 - 3x^2 - x^3$

E $y = x^3 - 3x^2 - 1$

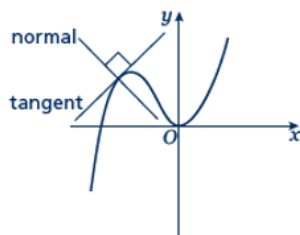
F $x + y = 5$

Hint

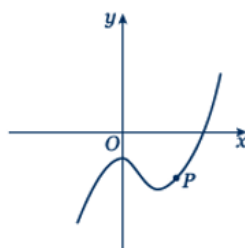
Find where each of the cubic equations cross the y-axis.

Here are six graphs.

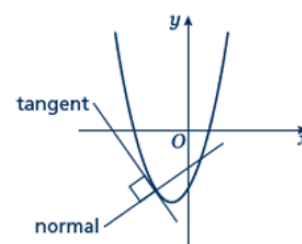
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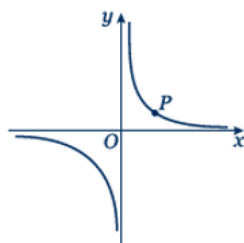
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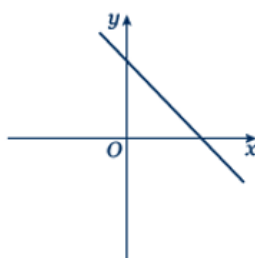
iii



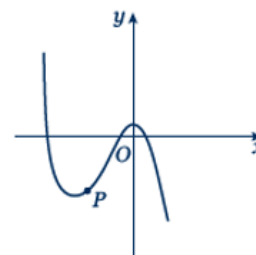
iv



v



vi



a Match each graph to its equation.

b Copy the graphs ii, iv and vi and draw the tangent and normal each at point P.

Sketch the following graphs

2 $y = 2x^3$

3 $y = x(x - 2)(x + 2)$

4 $y = (x + 1)(x + 4)(x - 3)$

5 $y = (x + 1)(x - 2)(1 - x)$

6 $y = (x - 3)^2(x + 1)$

7 $y = (x - 1)^2(x - 2)$

8 $y = \frac{3}{x}$

Hint: Look at the shape of $y = \frac{a}{x}$ in the second key point.

9 $y = -\frac{2}{x}$

Extend

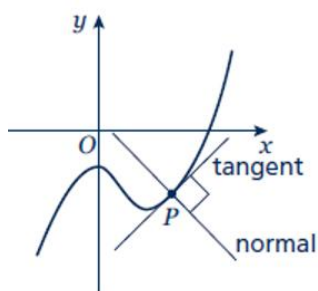
10 Sketch the graph of $y = \frac{1}{x+2}$

11 Sketch the graph of $y = \frac{1}{x-1}$

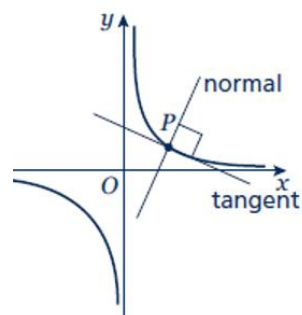
Answers

- 1 a i – C
 ii – E
 iii – B
 iv – A
 v – F
 vi – D

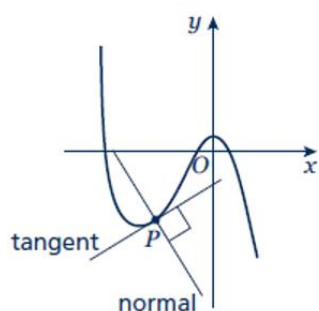
b ii



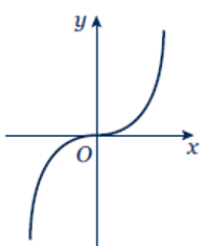
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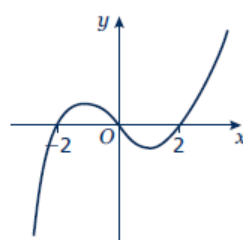
vi



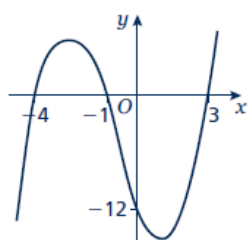
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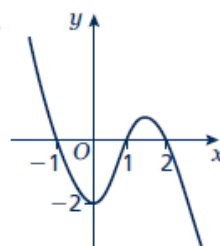
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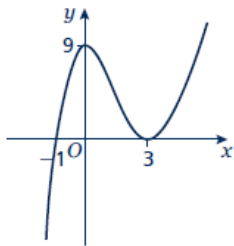
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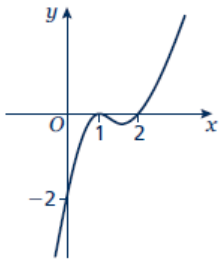
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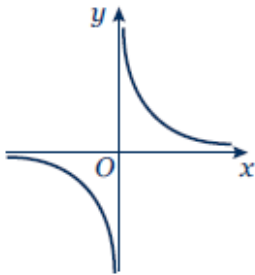
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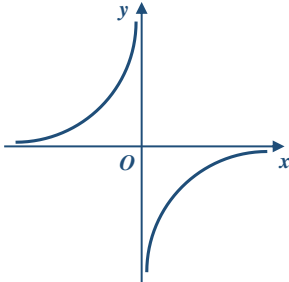
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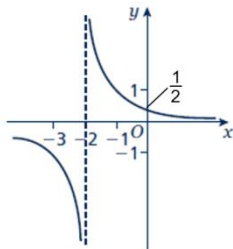
8



9



10



11

